

Abelian Zero Modes in Odd Dimensions

Gerald V. Dunne*

Department of Physics, University of Connecticut, Storrs, CT 06269

Hyunsoo Min†

Department of Physics, University of Seoul, Seoul 130-743, Korea

We show that the Loss-Yau zero modes of the 3d abelian Dirac operator may be interpreted in a simple manner in terms of a stereographic projection from a 4d Dirac operator with a constant field strength of definite helicity. This is an alternative to the conventional viewpoint involving Hopf maps from S^3 to S^2 . Furthermore, our construction generalizes in a straightforward way to any odd dimension. The number of zero modes is related to the Chern-Simons number in a nonlinear manner.

The behavior of quantized charged fermions in ultra-strong magnetic fields has applications in atomic physics [1], particle and condensed matter physics [2], and astrophysics [3]. Key properties determining stability are the existence of zero modes for the 3 dimensional Dirac operator, and the associated magnetic helicity. In studying the stability of atoms in magnetic fields, Loss and Yau [1] found the surprising result that the abelian Dirac operator in 3 dimensions, $\mathcal{D}_3 \equiv i\gamma_\mu (\partial/\partial x^\mu - iA_\mu)$, can have exact zero modes

$$i\gamma_\mu \left(\frac{\partial}{\partial x^\mu} - iA_\mu \right) \psi^{(0)} = 0 \quad (1)$$

for smooth, localized magnetic fields $\vec{B} = \vec{\nabla} \times \vec{A}$. In even dimensions there is a well-known relation between zero modes and the topology of gauge fields [4], but in odd dimensions, where the relevant index theorem is due to Callias [5], the situation is somewhat different, as the index is determined by the topology of the coupling to a Higgs field. In this Brief Report we present a simple new interpretation of the Loss-Yau zero-mode-supporting abelian gauge fields, and show that this construction generalizes to all odd dimensions.

Loss and Yau's simplest example [1] is the gauge field

$$\vec{A}_{LY} = \frac{3}{1+\vec{x}^2} \hat{N} \quad , \quad \hat{N} \equiv \frac{1}{1+\vec{x}^2} \begin{pmatrix} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix} \quad , \quad \vec{B}_{LY} = \frac{12}{(1+\vec{x}^2)^2} \hat{N} \quad , \quad (2)$$

for which the zero mode is

$$\psi_{LY}^{(0)} = \frac{4}{(1+\vec{x}^2)^{3/2}} (1 + i\gamma_\mu x_\mu) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad . \quad (3)$$

For this field, the Chern-Simons number, or magnetic helicity, is

$$\mathcal{N}_{LY}^{CS} \equiv \frac{1}{16\pi^2} \int d^3x \vec{A} \cdot \vec{B} = \frac{9}{16} \quad . \quad (4)$$

The associated magnetic field is plotted in Figure 1, showing the localized but non-trivial structure of the field. This basic example may be extended [1, 6] to fields with multiple zero modes:

$$\vec{A}_{LY} = \frac{(2k+3)}{1+\vec{x}^2} \hat{N} \quad ; \quad \vec{B}_{LY} = \frac{4(2k+3)}{(1+\vec{x}^2)^2} \hat{N} \quad ; \quad \mathcal{N}_{LY}^{CS} = \frac{(2k+3)^2}{16} \quad . \quad (5)$$

Here $k \geq 0$ is an integer, and this field has $(k+1)$ zero modes that can be expressed in terms of 3d spherical harmonics [1, 6].

These fields have since been discussed in terms of Hopf maps [6, 7], which are maps $\chi : S^3 \rightarrow S^2$ such that the magnetic field is

$$\vec{B}_H = \frac{2}{i} \frac{\vec{\nabla}\chi \times \vec{\nabla}\bar{\chi}}{(1+\chi\bar{\chi})^2} \quad . \quad (6)$$

*Electronic address: dunne@phys.uconn.edu

†Electronic address: hsmi@dirac.uos.ac.kr

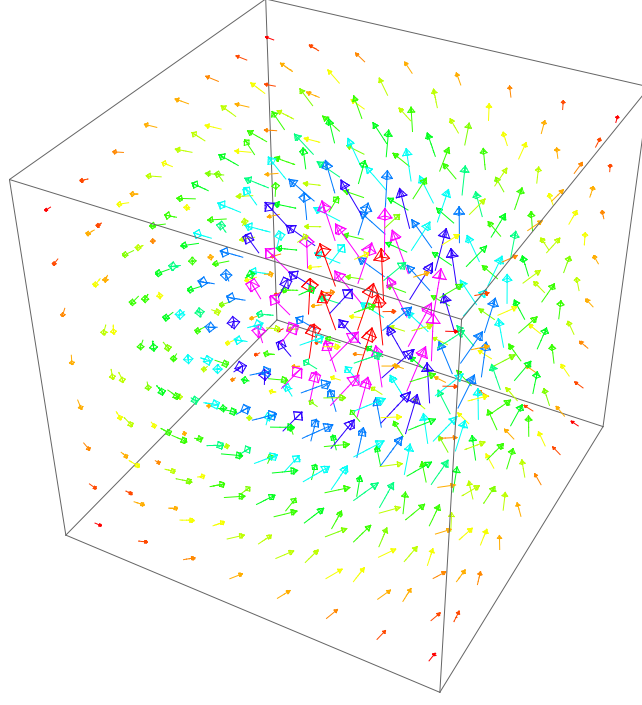


FIG. 1: Plot of the magnetic field vector \vec{B}_{LY} in (2). Note that the magnitude is highly localized around the origin, while the direction winds in a non-trivial manner.

Such a Hopf map can also be viewed as a map $\chi : \mathbf{R}^3 \rightarrow \mathbf{R}^2$, and the simplest example

$$\chi = \frac{(x_1 + ix_2)}{2x_3 - i(1 - \bar{x}^2)} \quad ; \quad \vec{B}_H = \frac{16}{(1 + \bar{x}^2)^2} \hat{N} = \frac{4}{3} \vec{B}_{LY} \quad ; \quad \mathcal{N}_H^{CS} = 1 \quad (7)$$

gives a magnetic field proportional to the Loss-Yau field in (2). Geometrically, \vec{B}_H is tangent to the closed curves in \mathbf{R}^3 given by $\chi = \text{constant}$. Erdős and Solovej [7] gave an elegant interpretation of these zero-mode-supporting gauge fields in terms of pull-backs (to \mathbf{R}^3) of 2 dimensional magnetic fields, and the 3 dimensional zero modes were related to the Aharonov-Casher zero modes in 2 dimensions [8]. Further results have been found in [6, 9], and these gauge fields have also been understood in terms of projections of non-abelian fields [10].

However, a number of questions remain. The fundamental mis-match of the coefficient [the factor 4/3 in (7)] does not have an elegant interpretation in the Hopf map language. In one picture, one introduces an additional "background" field with a correcting coefficient [6]; and in another picture [7], one includes a magnetic monopole of a particular strength at the centre of the S^2 to adjust the strength of the area form.

In this short note we present another characterization of these 3d abelian zero-mode-supporting gauge fields, in terms of four dimensional gauge fields of fixed helicity. This construction is extremely simple, and furthermore it generalizes naturally to other odd dimensions.

Our basic example [the analogue of (2)] is expressed for arbitrary odd dimension by a stereographic projection from $\mathbf{R}^{2n} \supset S^{2n-1} \rightarrow \mathbf{R}^{2n-1}$. We define coordinates x_μ ($\mu = 1, 2, \dots, 2n-1$) on \mathbf{R}^{2n-1} , and coordinates y_a ($a = 1, \dots, 2n$) on \mathbf{R}^{2n} . Consider a $2n$ -dimensional gauge field corresponding to a constant field strength, and such that the field has fixed helicity :

$$\begin{aligned} \mathcal{A}_a &= -\frac{\mathcal{F}}{2} (y_2, -y_1, y_4, -y_3, \dots, y_{2n-2}, -y_{2n-3}, -y_{2n}, y_{2n-1}) \\ &\equiv -\frac{\mathcal{F}}{2} J_{ab} y_b \end{aligned} \quad (8)$$

where the antisymmetric matrix $J = \text{diag}(i\sigma_2, \dots, i\sigma_2, -i\sigma_2)$. [The sign-flip in the last diagonal entry is a parity convention chosen to agree with the choice of Loss-Yau.] This gauge field is in Fock-Schwinger gauge: $y_a \mathcal{A}_a = 0$. Now restrict to S^{2n-1} by imposing the condition $y_a^2 = 1$, and stereographically project from S^{2n-1} to \mathbf{R}^{2n-1} via:

$$y_\mu = \frac{2x_\mu}{1 + \bar{x}^2} \quad , \quad y_{2n} = \frac{1 - \bar{x}^2}{1 + \bar{x}^2} \quad . \quad (9)$$

The projected $(2n - 1)$ -dimensional gauge field A_μ is [17]

$$A_\mu = \frac{\partial y_a}{\partial x_\mu} \mathcal{A}_a \quad (10)$$

One finds by a simple computation

$$\begin{aligned} A_i &= 2\mathcal{F} \left(\frac{-J_{ij}x_j + x_i x_{2n-1}}{(1 + \vec{x}^2)^2} \right) \quad , \quad i = 1, 2, \dots, 2n - 2 \\ A_{2n-1} &= \mathcal{F} \left(\frac{1 - \vec{x}^2 + 2x_{2n-1}^2}{(1 + \vec{x}^2)^2} \right) \end{aligned} \quad (11)$$

When $n = 2$ (i.e., a 3 dimensional gauge field A_μ) this reproduces precisely the form of the original Loss-Yau gauge field [1] in (2), although the coefficient \mathcal{F} is not yet determined.

The coefficient \mathcal{F} is fixed by an explicit construction of the zero mode, directly from the zero mode equation (1). Straightforward Dirac algebra manipulations [1, 14] show that the gauge field in (1) can be expressed in terms of the zero mode $\psi_{(0)}$

$$A_\mu = \frac{\partial_\nu \left(\psi_{(0)}^\dagger \sigma_{\mu\nu} \psi_{(0)} \right)}{\psi_{(0)}^\dagger \psi_{(0)}} + i \frac{\left(\partial_\mu \psi_{(0)}^\dagger \psi_{(0)} - \psi_{(0)}^\dagger \partial_\mu \psi_{(0)} \right)}{2\psi_{(0)}^\dagger \psi_{(0)}} \quad (12)$$

where $\sigma_{\mu\nu} = \frac{1}{4i}[\gamma_\mu, \gamma_\nu]$, with the constraint $\partial_\mu \left(\psi_{(0)}^\dagger \gamma_\mu \psi_{(0)} \right) = 0$.

For our gauge field (11) we postulate the (un-normalized) 2^{n-1} component Dirac zero mode spinor

$$\psi_{(0)} = \frac{(\mathbf{1} + i\gamma_\mu x_\mu)}{(1 + \vec{x}^2)^{n-1/2}} (1, \dots, 1 | 0, \dots, 0)^T \quad (13)$$

Then the resulting gauge field constructed from (12) in $(2n - 1)$ dimensions is

$$A_i = 2(2n - 1) \left(\frac{-J_{ij}x_j + x_i x_{2n-1}}{(1 + \vec{x}^2)^2} \right) \quad , \quad i = 1, 2, \dots, 2n - 2 \quad (14)$$

$$A_{2n-1} = (2n - 1) \left(\frac{1 - \vec{x}^2 + 2x_{2n-1}^2}{(1 + \vec{x}^2)^2} \right) \quad (15)$$

which is precisely the same as (11), but now the overall coefficient has been fixed to be $\mathcal{F} = 2n - 1$. When $n = 2$ (corresponding to 3 dimensions) this reproduces the original Loss-Yau gauge field in (2). Note that the zero mode (13) is normalizable in all odd dimensions $d = 2n - 1 \geq 3$.

The degeneracy of the abelian zero modes can be deduced by group theoretic arguments for spinors in arbitrary dimensions. Define $2^{n-1} \times 2^{n-1}$ Dirac matrices γ_μ ($\mu = 1, \dots, (2n - 1)$) for \mathbf{R}^{2n-1} , and $2^n \times 2^n$ Dirac matrices Γ_a ($a = 1, \dots, 2n$) for \mathbf{R}^{2n} . These can be related as

$$\Gamma_\mu = \begin{pmatrix} 0 & i\gamma_\mu \\ -i\gamma_\mu & 0 \end{pmatrix} \quad ; \quad \Gamma_{2n} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad ; \quad \Gamma_{2n+1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad (16)$$

Then the spin matrices in \mathbf{R}^{2n} may be block-decomposed as

$$\Sigma_{ab} \equiv \frac{1}{4i}[\Gamma_a, \Gamma_b] = \begin{pmatrix} \Sigma_{ab}^+ & 0 \\ 0 & \Sigma_{ab}^- \end{pmatrix} \quad (17)$$

where $\Sigma_{\mu\nu}^\pm = \sigma_{\mu\nu} \equiv \frac{1}{4i}[\gamma_\mu, \gamma_\nu]$, and $\Sigma_{\mu, 2n}^\pm = \pm \frac{1}{2} \gamma_\mu$. We also define the $2n$ dimensional angular momentum generators

$$L_{ab} \equiv -i \left(y_a \frac{\partial}{\partial y_b} - y_b \frac{\partial}{\partial y_a} \right) \quad (18)$$

Then a canonical result of stereographic projection of the *free* Dirac equation from S^{2n-1} (defined by $y_a y_a = 1$) to \mathbf{R}^{2n-1} is that

$$\left(\frac{1 + \vec{x}^2}{2} \right)^{2n-1} i\gamma_\mu \frac{\partial}{\partial x_\mu} = \left(\frac{1 + \vec{x}^2}{2} \right)^{n-3/2} V^\dagger \left[\Sigma_{ab}^+ L_{ab} + \left(n - \frac{1}{2} \right) \mathbf{1} \right] V \left(\frac{1 + \vec{x}^2}{2} \right)^{n-3/2} \quad (19)$$

where $V \equiv \frac{1}{\sqrt{2}}(1 + i\gamma_\mu x_\mu)$.

Now we observe that for the $2n$ dimensional gauge field \mathcal{A}_a defined in (10) and the $(2n-1)$ -dimensional gauge field A_μ defined in (15), this projection property of the free Dirac equation is maintained once the gauge field is included:

$$\left(\frac{1+\vec{x}^2}{2}\right)^{2n-1} i\gamma_\mu \left(\frac{\partial}{\partial x_\mu} - iA_\mu\right) = \left(\frac{1+\vec{x}^2}{2}\right)^{n-3/2} V^\dagger \left[\Sigma_{ab}^+ \mathcal{L}_{ab} + \left(n - \frac{1}{2}\right) \mathbf{1}\right] V \left(\frac{1+\vec{x}^2}{2}\right)^{n-3/2} \quad (20)$$

where $\mathcal{L}_{ab} \equiv L_{ab} + (y^a \mathcal{A}_b - y^b \mathcal{A}_a)$. Thus, the zero-mode equation on \mathbf{R}^{2n-1} can be lifted to a zero-mode equation on S^{2n-1} , where the solutions can be written in terms of the spinor spherical harmonics in \mathbf{R}^{2n} .

To illustrate this explicitly we consider the $n=2$ case. The 4-dimensional gauge field may be written as $\mathcal{A}_a = -\mathcal{F}/2 \bar{\eta}_{ab}^3 y_b$, where $\bar{\eta}_{ab}^3$ is the 3rd isospin component of the standard 4-dim. 't Hooft tensor [12, 15]. The 4-component spinor zero mode $\psi_{(0)}$ in \mathbf{R}^4 may be written in terms of a 2-component spinor u of definite (we choose positive) helicity:

$$\psi_{(0)} = \begin{pmatrix} u \\ 0 \end{pmatrix} \quad (21)$$

Then using (20) the zero mode equation (1) becomes an algebraic equation

$$[\Sigma_{ab}^+ \mathcal{L}_{ab} + 3/2] u = \left(4\vec{S} \cdot \vec{L} + 3/2 - \frac{\mathcal{F}}{2} \sigma_3\right) u = 0 \quad (22)$$

where $u = ((1+\vec{x}^2)/2)^{1/2} V \phi$, and where \vec{S} and \vec{L} are angular momentum operators, of spin $1/2$ and l ($=$ half integer), respectively. We define the total angular momentum $\vec{J} = \vec{S} + \vec{L}$, and use the spinor spherical harmonics [16]

$$u^{(\pm)} = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \pm \sqrt{l+1/2 \pm M} Y_{m,M-1/2}^l \\ \sqrt{l+1/2 \mp M} Y_{m,M+1/2}^l \end{pmatrix} \quad (23)$$

with $j = l \pm 1/2$, $-j \leq M \leq j$, and $-l \leq m \leq l$. Then

$$4\vec{S} \cdot \vec{L} u^{(\pm)} = \begin{Bmatrix} 2l \\ -2l-2 \end{Bmatrix} u^{(\pm)}. \quad (24)$$

Thus a zero mode can only occur when $j = M = l + 1/2$ and

$$2l + 3/2 - \frac{\mathcal{F}}{2} = 0 \quad \rightarrow \quad \mathcal{F} = 4l + 3 \quad (25)$$

for arbitrary value of $-l \leq m \leq l$. In this case $u^{(+)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} Y_{m,l}^l$. Recalling that $l = k/2$ is a half-integer, we arrive at the general Loss-Yau case (5). Furthermore, the degeneracy is simply given by $2l+1 = k+1$, as found by Loss-Yau in [1]. This construction makes it clear why the zero mode degeneracy factor is linear in the integer k , while the Chern-Simons number is quadratic in k . An analogous construction is clearly possible in higher dimensions using generalized spinor spherical harmonics, but we do not present the details here.

To conclude, we have given a simple new interpretation of the Loss-Yau abelian zero-mode-supporting gauge fields in three dimensions, and have extended the construction to obtain new zero-mode-supporting abelian gauge fields in other odd dimensions. An interesting outstanding problem is the possibility of including an interaction with a scalar field, which might shed light on the possible relation of these fields to the Callias index theorem [5].

GD thanks the US DOE for support through grant DE-FG02-92ER40716, and HM thanks the UConn Research Foundation and the USeoul Research Foundation for grants.

-
- [1] M. Loss and H-T. Yau, "Stability of Coulomb Systems with Magnetic Fields", Comm. Math. Phys. **104**, 283 (1986).
 [2] V. P. Gusynin, V. A. Miransky and I. A. Shovkovy, "Dynamical chiral symmetry breaking by a magnetic field in QED," Phys. Rev. D **52**, 4747 (1995) [arXiv:hep-ph/9501304].

- [3] G. B. Field and S. M. Carroll, “Cosmological magnetic fields from primordial helicity,” *Phys. Rev. D* **62**, 103008 (2000) [arXiv:astro-ph/9811206].
- [4] S. B. Trieman, R. Jackiw, B. Zumino and E. Witten (Eds.), *Current Algebras and Anomalies* (Princeton Univ. Press, 1985).
- [5] C. Callias, “Index Theorems On Open Spaces,” *Commun. Math. Phys.* **62**, 213 (1978).
- [6] C. Adam, B. Muratori and C. Nash, “Zero modes of the Dirac operator and the Seiberg-Witten equations in three dimensions,” *Phys. Rev. D* **60**, 125001 (1999) [arXiv:hep-th/9903040]; “Degeneracy of zero modes of the Dirac operator in three dimensions,” *Phys. Lett. B* **485**, 314 (2000) [arXiv:hep-th/9910139]; “Multiple zero modes of the Dirac operator in three dimensions,” *Phys. Rev. D* **62**, 085026 (2000) [arXiv:hep-th/0001164].
- [7] L. Erdős and J. P. Solovej, “The kernel of Dirac operators on S^3 and R^3 ,” *Rev. Math. Phys.* **13**, 1247 (2001). [arXiv:math-ph/0001036].
- [8] Y. Aharonov and A. Casher, “The Ground State Of A Spin 1/2 Charged Particle In A Two-Dimensional Magnetic Field,” *Phys. Rev. A* **19**, 2461 (1979).
- [9] D. M. Elton, “New examples of zero modes”, *J. Phys. A* **33**, 7297 (2000), “The Local Structure of Zero Mode Producing Magnetic Potentials”, *Comm. Math. Phys.* **229**, 121 (2002).
- [10] R. Jackiw and S. Y. Pi, “Creation and evolution of magnetic helicity,” *Phys. Rev. D* **61**, 105015 (2000) [arXiv:hep-th/9911072].
- [11] S. L. Adler, “Massless Electrodynamics On The Five-Dimensional Unit Hypersphere: An Amplitude - Integral Formulation,” *Phys. Rev. D* **8**, 2400 (1973) [Erratum-ibid. *D* **15**, 1803 (1977)].
- [12] R. Jackiw and C. Rebbi, “Conformal properties of a Yang-Mills pseudoparticle,” *Phys. Rev. D* **14**, 517 (1976), “Spinor analysis of Yang-Mills theory,” *Phys. Rev. D* **16**, 1052 (1977).
- [13] E. B. Bogomolny and Yu. A. Kubyshin, “Asymptotical Estimates For Graphs With A Fixed Number Of Fermionic Loops In Quantum Electrodynamics. The Extremal Configurations With The Symmetry Group $O(2) \times O(3)$,” *Sov. J. Nucl. Phys.* **35**, 114 (1982) [*Yad. Fiz.* **35**, 202 (1982)].
- [14] H. S. Booth, G. Legg and P. D. Jarvis, “Algebraic solution for the vector potential in the Dirac equation,” *J. Phys. A* **34**, 5667 (2001) [arXiv:hep-th/0104216].
- [15] G. 't Hooft, “Computation of the quantum effects due to a four-dimensional pseudoparticle,” *Phys. Rev. D* **14**, 3432 (1976) [Erratum-ibid. *D* **18**, 2199 (1978)].
- [16] A. Pais, “Spherical spinors in a Euclidean 4-space”, *Proc. Nat. Acad. Sci.* **40**, 835 (1954).
- [17] Such gauge field projections have been studied extensively for projections to \mathbf{R}^4 [11, 12, 13]; here we consider analogous projections to odd dimensional spaces.